

THE DONALDSON-THOMAS THEORY OF $K3 \times E$ VIA THE TOPOLOGICAL VERTEX.

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ABSTRACT. Oberdieck and Pandharipande conjectured [9] that the curve counting invariants of $S \times E$, the product of a $K3$ surface and an elliptic curve, is given by minus the reciprocal of the Igusa cusp form of weight 10. For a fixed primitive curve class in S of square $2h - 2$, their conjecture predicts that the corresponding partition functions are given by meromorphic Jacobi forms of weight -10 and index $h - 1$. We prove their conjecture for primitive classes of square -2 and of square 0 .

Our computation uses reduced Donaldson-Thomas invariants which are defined as the Behrend function weighted Euler characteristics of the quotient of the Hilbert scheme of curves in $S \times E$ by the action of E . Our technique is a mixture of motivic and toric methods (developed with Kool in [5]) which allows us to express the partition functions in terms of the topological vertex and subsequently in terms of Jacobi forms.

1. OVERVIEW

Let $X = S \times E$ where S is a $K3$ surface and E is an elliptic curve. In [9], Oberdieck and Pandharipande conjectured that the partition function for the curve counting invariants of X is given by $-1/\chi_{10}$, minus the reciprocal of the Igusa cusp form of weight 10. The relevant curve counting invariants include modified versions of Gromov-Witten invariants and stable pairs invariants. In this paper, we define modified Donaldson-Thomas invariants of X . Our definition is given by taking the Behrend function weighted Euler characteristic of the quotient of the Hilbert scheme of curves on X by the action of the elliptic curve. Our invariants are expected to be equal to the invariants defined via stable pairs in [9].

We employ an approach to computing these invariants which uses a mixture of motivic and toric methods (technology developed with M. Kool in [5]). We show that these methods yield complete computations for the partition functions of X in the case where S is $K3$ surface with a primitive curve class of square -2 or of square 0 . The resulting partition functions are given by the Jacobi forms $F^{-2}\Delta^{-1}$ and $-24\wp\Delta^{-1}$ respectively where F is a Jacobi theta function, Δ is the discriminant modular form, and \wp is the Weierstrass \wp function. This agrees with the prediction from by the Oberdieck-Pandharipande conjecture thus proving their conjecture for primitive curve classes in the $K3$ of square -2 or 0 .

Our general computational strategy is the following. Donaldson-Thomas invariants are given by weighted Euler characteristics of Hilbert schemes. We stratify the Hilbert scheme using the geometric support of the curves and we compute Euler characteristics of strata separately. Many of the strata acquire actions of E or \mathbb{C}^* (that were not present globally) and we restrict to the fixed point loci. We are able to further stratify the fixed point loci and those strata sometimes acquire further actions. Iterating this strategy, we reduce the computation to subschemes which are formally locally given by monomial ideals. These are counted using the topological vertex. New identities for the topological vertex lead to

closed formulas. We use the Hall algebra techniques of Joyce-Song and Bridgeland [7, 3] to incorporate the Behrend function into this strategy.

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2. DEFINITIONS AND CONJECTURES

Let X be an arbitrary non-singular Calabi-Yau threefold over \mathbb{C} . One can define Donaldson-Thomas curve counting invariants by taking weighted Euler characteristics of the Hilbert scheme of curves in X . Let

$$\mathrm{Hilb}^{\beta,n}(X) = \{Z \subset X : [Z] = \beta \in H_2(X), n = \chi(\mathcal{O}_Z)\}$$

be the Hilbert scheme of proper subschemes of X with fixed homology class and holomorphic Euler characteristic.

The Behrend function is a integer-valued constructible function associated to any scheme over \mathbb{C} . One can define the Donaldson-Thomas invariants of X by

$$DT_{\beta,n}(X) = \sum_{k \in \mathbb{Z}} k \cdot e(\nu^{-1}(k))$$

where

$$\nu : \mathrm{Hilb}^{\beta,n}(X) \rightarrow \mathbb{Z}$$

is the Behrend function [1].

It will be notationally convenient to treat an Euler characteristic weighted by a constructible function as a Lebesgue integral, where the measurable sets are constructible sets, the measurable functions are constructible functions, and the measure of a set is given by its Euler characteristic. In this language, one writes

$$DT_{\beta,n}(X) = \int_{\mathrm{Hilb}^{\beta,n}(X)} \nu \, de.$$

For proper X , $DT_{\beta,n}(X)$ as defined above is invariant under deformations of X .

We now consider

$$X = S \times E$$

where S is a non-singular $K3$ surface with a primitive curve class β of square

$$\beta^2 = 2h - 2.$$

We call h the *genus* of the $K3$ surface. Let

$$\beta + dE \in H_2(X)$$

denote the class $i_{S*}(\beta) + i_{E*}(d[E])$ where $i_S : S \rightarrow X$ and $i_E : E \rightarrow X$ are the inclusions obtained from choosing points $s \in S$ and $e \in E$.

The Donaldson-Thomas invariants $DT_{\beta+dE,n}(X)$ are all zero. This can be seen in two different ways:

- (1) The action of E on $\text{Hilb}^{\beta+dE,n}(X)$ is fixed point free, consequently its (Behrend function¹ weighted) Euler characteristic is zero.
- (2) There exists deformations of S which make β non-algebraic. Under this deformation, the Hilbert scheme $\text{Hilb}^{\beta+dE,n}(X)$ becomes empty. Since $DT_{\beta+dE,n}(X)$ is deformation invariant it must be zero.

Remarkably, the above two issues can be solved simultaneously by taking the weighted Euler characteristic of the quotient of the Hilbert scheme.

Definition 2.1. *The reduced Donaldson-Thomas invariants of X are defined by*

$$DT_{\beta+dE,n}(X) = \int_{\text{Hilb}^{\beta+dE,n}(X)/E} \nu \, de$$

where $\nu : \text{Hilb}^{\beta+dE,n}(X)/E \rightarrow \mathbb{Z}$ is the Behrend function of the quotient. Note that we denote the reduced invariants with the sans serif font DT , while the ordinary invariants have the ordinary font DT .

Conjecture 2.2. *The number $DT_{\beta+dE,n}(X)$ is invariant under deformations of X which keep the class $\beta + dE$ algebraic.*

Proof sketch: The Hilbert scheme $\text{Hilb}^{\beta+dE,n}(X)$ admits a (-1) -shifted symplectic structure coming from viewing it as a moduli space of rank 1 sheaves on X with trivialized determinant [12]. Taking the (-1) -symplectic quotient of the Hilbert scheme by the action of E yields a (-1) -symplectic space whose underlying space is $\text{Hilb}^{\beta+dE,n}(X)/E$ (the moment map affects the derived structure, but not the classical space). As with any (-1) -shifted symplectic structure, this shifted symplectic structure gives rise to a symmetric obstruction theory whose associated virtual class has degree equal to the Behrend function weighted Euler characteristic of underlying scheme. The effect of taking the zeros of the moment map in the symplectic quotient construction is to remove from the obstruction space those obstructions to deforming the class β to a non-algebraic class. Note that these obstructions are dual to the deformations of a subscheme given by the action of E . The resulting virtual class on $\text{Hilb}^{\beta+dE,n}(X)/E$ should be invariant under deformations preserving the algebraicity of β .

Up to deformation, a curve class on a $K3$ surface is determined by its square and divisibility, so by our assumption that β is primitive, it only depends on h up to deformation. We thus streamline the notation by writing:

$$DT_{h,d,n}(X) := DT_{\beta+dE,n}(X)$$

and we also write

$$\text{Hilb}^{h,d,n}(X) := \text{Hilb}^{\beta+dE,n}(X).$$

We also consider the related (but not *a priori* deformation invariant) quantity given by unweighted Euler characteristics.

$$\widehat{DT}_{h,d,n}(X) = \int_{\text{Hilb}^{h,d,n}(X)/E} 1 \, de.$$

¹The value of the Behrend function at a closed point of a scheme only depends on the local ring of that point, therefore the Behrend function of a scheme is invariant under any group action.

We define partition functions as follows

$$\begin{aligned}
\mathrm{DT}(X) &= \sum_{h=0}^{\infty} \mathrm{DT}_h(X) \tilde{q}^{h-1} \\
&= \sum_{\substack{h,d \geq 0 \\ n \in \mathbb{Z}}} \mathrm{DT}_{h,d,n}(X) \tilde{q}^{h-1} q^{d-1} (-p)^n \\
\widehat{\mathrm{DT}}(X) &= \sum_{h=0}^{\infty} \widehat{\mathrm{DT}}_h(X) \tilde{q}^{h-1} \\
&= \sum_{\substack{h,d \geq 0 \\ n \in \mathbb{Z}}} \widehat{\mathrm{DT}}_{h,d,n}(X) \tilde{q}^{h-1} q^{d-1} p^n
\end{aligned}$$

We remark that our convention for the \tilde{q} and q variables is the opposite from Oberdieck and Pandharipande's, however there is a conjectural symmetry $\tilde{q} \leftrightarrow q$ and so this difference should not be seen in the formulas. To be precise, the Donaldson-Thomas version of Oberdieck and Pandharipande's conjecture is the following.

Conjecture 2.3. *Let χ_{10} be the Igusa cusp form of weight 10, then*

$$\mathrm{DT}(X) = -\frac{1}{\chi_{10}}.$$

Explicitly, we can write

$$\chi_{10}(p, q, \tilde{q}) = pq\tilde{q} (1 - p^{-1})^2 \prod_{n \in \mathbb{Z}} \prod_{(d,h) > (0,0)} (1 - p^n q^d \tilde{q}^h)^{c(4dh - n^2)}$$

where the integers $c(k)$ are given as the coefficients of Z , the elliptic genus of the $K3$ surface:

$$Z(p, q) = -24\wp F^2 = \sum_{n \in \mathbb{Z}} \sum_{d \geq 0} c(4d - n^2) p^n q^d.$$

Here F is a Jacobi theta function and \wp is the Weierstrass \wp -function, namely

$$-F^{-2} = \frac{p}{(1-p)^2} \prod_{m=1}^{\infty} \frac{(1-q^m)^4}{(1-pq^m)^2 (1-p^{-1}q^m)^2}$$

and

$$\wp = \frac{1}{12} + \frac{p}{(1-p)^2} + \sum_{d=1}^{\infty} \left(\sum_{k|d} k(p^k + p^{-k} - 2) \right) q^d.$$

Expanding $-\chi_{10}^{-1}$ as a series in \tilde{q} , one obtains predictions for each $\mathrm{DT}_h(X)$ in terms of Jacobi forms of weight -10 and index $h-1$ (see [9, page 10]). The main result of this paper is the following theorem.

Theorem 2.4. *The Oberdieck-Pandharipande conjecture holds for $K3$ surfaces with a primitive, generic², curve class of square -2 or 0 . Namely, the series $\mathrm{DT}_h(X)$ for $h = 0$*

²If we assume Conjecture 2.2, then this genericity assumption can be removed.

and $h = 1$ is given by the following Jacobi forms

$$\begin{aligned} \mathrm{DT}_0(X) &= \frac{1}{F^2 \Delta}, \\ \mathrm{DT}_1(X) &= -24 \frac{\wp}{\Delta}. \end{aligned}$$

Explicitly, the series are given by:

$$\begin{aligned} \mathrm{DT}_0(X) &= \frac{-pq^{-1}}{(1-p)^2} \prod_{m=1}^{\infty} (1-q^m)^{-20} (1-pq^m)^{-2} (1-p^{-1}q^m)^{-2} \\ \mathrm{DT}_1(X) &= -24q^{-1} \prod_{m=1}^{\infty} (1-q^m)^{-24} \left(\frac{1}{12} + \frac{p}{(1-p)^2} + \sum_{d=1}^{\infty} \sum_{k|d} k (p^k - 2 + p^{-k}) q^d \right) \end{aligned}$$

3. PRELIMINARIES AND NOTATION.

Our aim is to compute $\mathrm{DT}_h(X)$ for $h = 0$ and $h = 1$. We begin by computing $\widehat{\mathrm{DT}}_h(X)$ and then discuss how to modify the argument to include the Behrend function in section 6.

Euler characteristic is motivic: it defines a homomorphism from $K_0(\mathrm{Var}_{\mathbb{C}})$, the Grothendieck group of varieties over \mathbb{C} , to the integers. We define

$$\mathrm{Hilb}^{h, \bullet, \bullet}(X)/E = \sum_{n,d} \left[\mathrm{Hilb}^{h,d,n}(X)/E \right] p^n q^d$$

which we regard as an element in $K_0(\mathrm{Var}_{\mathbb{C}})((p))[[q]]$. We will use this convention throughout:

Convention 3.1. When an index is replaced by a bullet, we will sum over the index, multiplying by the appropriate variable.

We see that with our notation

$$\widehat{\mathrm{DT}}_h(X) = q^{-1} e \left(\mathrm{Hilb}^{h, \bullet, \bullet}(X)/E \right).$$

Definition 3.2. Let p_S and p_E be the projections of $X = S \times E$ onto each factor and let $C \subset X$ be an irreducible curve. We say that C is **vertical** if $p_E : C \rightarrow E$ is degree zero and we say C is **horizontal** if $p_S : C \rightarrow S$ is degree zero. If both maps are of non-zero degree, we say C is **diagonal**. See Figure 1.

We will assume that $X = S \times E$ where S is generic among $K3$ surfaces admitting a primitive class β of square $2h - 2$. In particular, β is an irreducible class.

Since β is an irreducible class, any subscheme Z corresponding to a point in $\mathrm{Hilb}^{h,d,n}(X)$ must have a unique component $C_0 \subset Z$ which is either a vertical or a diagonal curve with all other curve components of Z being horizontal. Subschemes with C_0 diagonal cannot deform to subschemes with C_0 horizontal and so we get a decomposition of the Hilbert scheme into disjoint components corresponding to subschemes with vertical and diagonal components respectively:

$$\mathrm{Hilb}^{h,d,n}(X) = \mathrm{Hilb}_{\mathrm{vert}}^{h,d,n}(X) \sqcup \mathrm{Hilb}_{\mathrm{diag}}^{h,d,n}(X)$$

Diagonal curves do not appear in the $h = 0$ case, but do occur for $h \geq 1$.

FIGURE 1. A vertical curve (orange) contained in the slice $S \times \{x_0\}$ (light grey), a diagonal curve (pink), and two horizontal curves (green).

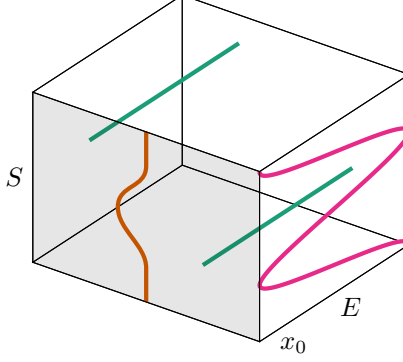
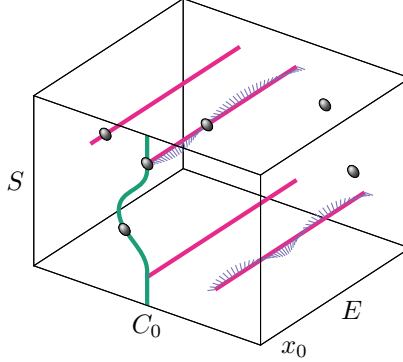


FIGURE 2. Subschemes in $S \times E$ up to translation. Horizontal curves (pink) can have nilpotent thickenings (blue), and there can be embedded and floating points (gray). The unique vertical curve C_0 (green) lies in $S \times \{x_0\}$ and is generically reduced.



4. COMPUTING $\widehat{\text{DT}}_h(X)$ IN THE CASE $h = 0$.

We now consider the case where $h = 0$. The $K3$ surface S has a single curve $C_0 \cong \mathbb{P}^1$ in the class β . There are no diagonal curves since such a curve would have geometric genus 0 but also admit a non-constant map to E .

We fix a base point $x_0 \in E$. We can fix a slice for the action of E on $\text{Hilb}^{0,d,n}(X)$ by requiring that the unique vertical curve lies in $S \times \{x_0\}$. We denote the slice with the subscript “fixed”.

$$\text{Hilb}^{0,d,n}(X)/E \cong \text{Hilb}_{\text{fixed}}^{0,d,n}(X) \subset \text{Hilb}^{0,d,n}(X).$$

The points in $\text{Hilb}_{\text{fixed}}^{0,d,n}(X)$ correspond to subschemes $Z \subset X$ given by unions of the curve $C_0 \times \{x_0\}$ with horizontal curves whose support is of the form $\{\text{points} \times E\}$, but may have nilpotent thickenings. The subscheme Z also potentially has embedded points as well as zero dimensional components away from the curve support (see Figure 2).

As a consequence of the above geometric description, we see that any such subscheme is a disjoint union of a subscheme of $\widehat{X}_{C_0 \times E}$, the formal neighborhood of $C_0 \times E$ in X ,

and $X - (C_0 \times E)$. This leads to a decomposition of the Hilbert scheme into strata given by products of Hilbert schemes of subschemes of $\widehat{X}_{C_0 \times E}$ and subschemes of $X - (C_0 \times E)$. Using our bullet convention, this can be efficiently expressed as follows.

$$(1) \quad \text{Hilb}^{0, \bullet, \bullet}(X)/E = \text{Hilb}^{0, \bullet, \bullet}(X - (C_0 \times E)) \cdot \text{Hilb}_{\text{fixed}}^{0, \bullet, \bullet}(\widehat{X}_{C_0 \times E})$$

where as before the subscript “fixed” indicates that we are restricting to the sublocus

$$\text{Hilb}_{\text{fixed}}^{0, d, n}(\widehat{X}_{C_0 \times E}) \subset \text{Hilb}^{0, d, n}(\widehat{X}_{C_0 \times E}) \subset \text{Hilb}^{0, d, n}(X)$$

parameterizing subschemes where the unique vertical curve is $C_0 \times \{x_0\}$.

Note that d (the degree in the E direction) and n (the holomorphic Euler characteristic) are both additive under the disjoint union which allows us to express the decomposition as a product of Grothendieck group valued power series as above. Taking Euler characteristics of the above series, we find

$$(2) \quad q \widehat{\text{DT}}_0(X) = e(\text{Hilb}^{0, \bullet, \bullet}(X - C_0 \times E)) \cdot e(\text{Hilb}_{\text{fixed}}^{0, \bullet, \bullet}(\widehat{X}_{C_0 \times E})).$$

Note that the action of E on $X - C_0 \times E$ induces an action on $\text{Hilb}^{0, d, n}(X - C_0 \times E)$. This “new” E action is possible because the “fixed” condition lives entirely in the $\text{Hilb}_{\text{fixed}}^{0, d, n}(\widehat{X}_{C_0 \times E})$ factors (which do not have E actions).

The Euler characteristic of a scheme with a free E action is trivial and so

$$e(\text{Hilb}^{0, d, n}(X - C_0 \times E)) = e(\text{Hilb}^{0, d, n}(X - C_0 \times E)^E).$$

The E -fixed locus $\text{Hilb}^{0, d, n}(X - C_0 \times E)^E$ parameterizes subschemes which are invariant under the E action. Such subschemes are of the form $Z \times E$ where $Z \subset S - C_0$ is a zero-dimensional subscheme of length d . Such subschemes have $n = \chi(\mathcal{O}_{Z \times E}) = 0$ and so

$$(3) \quad \begin{aligned} e(\text{Hilb}^{0, \bullet, \bullet}(X - C_0 \times E)^E) &= e\left(\sum_{d=0}^{\infty} \text{Hilb}^d(S - C_0) q^d\right) \\ &= \prod_{m=1}^{\infty} (1 - q^m)^{-22}. \end{aligned}$$

Here we have used Göttsche’s formula for the Euler characteristics of Hilbert schemes of points of surfaces; the 22 appearing in the exponent is the Euler characteristic of the surface $S - C_0$.

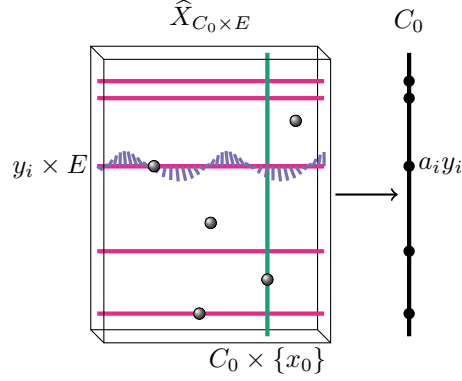
To compute $e(\text{Hilb}_{\text{fixed}}^{0, \bullet, \bullet}(\widehat{X}_{C_0 \times E}))$, we begin by noting that there is a morphism

$$\rho_d : \text{Hilb}_{\text{fixed}}^{0, d, n}(\widehat{X}_{C_0 \times E}) \rightarrow \text{Sym}^d(C_0)$$

given by the intersection (with multiplicity) of the horizontal components of a curve with the vertical curve C_0 . In other words, a scheme whose curve support is $C_0 \cup_i (y_i \times E)$ with multiplicity a_i along $y_i \times E$ is mapped to $\sum_i a_i y_i \in \text{Sym}^d(C_0)$ (see Figure 3).

We may compute the Euler characteristic of $\text{Hilb}_{\text{fixed}}^{0, \bullet, \bullet}(\widehat{X}_{C_0 \times E})$ by computing the Euler characteristic of $\text{Sym}^d(C_0)$, weighted by the constructible function given by the Euler characteristic of the fibers of ρ_d . In other words,

FIGURE 3. The map $\rho_d : \text{Hilb}_{\text{fixed}}^{0,d,n}(\widehat{X}_{C_0 \times E}) \rightarrow \text{Sym}^d(C_0)$ records the location and multiplicity of the horizontal curve components.



$$\begin{aligned} e\left(\text{Hilb}_{\text{fixed}}^{0,d,n}(\widehat{X}_{C_0 \times E})\right) &= \int_{\text{Hilb}_{\text{fixed}}^{0,d,n}(\widehat{X}_{C_0 \times E})} 1 \, de \\ &= \int_{\text{Sym}^d C_0} (\rho_d)_*(1) \, de. \end{aligned}$$

Writing

$$\text{Sym}^\bullet C_0 = \sum_{d=0}^{\infty} \text{Sym}^d C_0 q^d$$

and extending the integration to the \bullet notation in the obvious way, we get

$$(4) \quad e\left(\text{Hilb}_{\text{fixed}}^{0,\bullet,\bullet}(\widehat{X}_{C_0 \times E})\right) = \int_{\text{Sym}^\bullet C_0} \rho_*(1) \, de$$

where the constructible function $\rho_*(1)$ takes values in $\mathbb{Z}((p))$ and is given by

$$\rho_*(1)\left(\sum_i a_i y_i\right) = e\left(\rho^{-1}\left(\sum_i a_i y_i\right)\right).$$

We will prove that $\rho_*(1)$ only depends on the multiplicities of the points in the symmetric product, not their location.

Proposition 4.1. *There exists a universal series $F(a) \in \mathbb{Z}[[p]]$ such that the constructible function $\rho_*(1)$ is given by*

$$\rho_*(1)\left(\sum a_i y_i\right) = \frac{p}{(1-p)^2} \prod_i F(a_i).$$

Deferring the proof of the proposition for the moment, we apply the following lemma regarding weighted Euler characteristics of symmetric products.

Lemma 4.2. *Let T be a scheme and let $\text{Sym}^\bullet(T) = \sum_{d=0}^{\infty} \text{Sym}^d(T) q^d$. Suppose that G is a constructible function on $\text{Sym}^d(T)$ of the form $G(\sum_i a_i y_i) = \prod_i g(a_i)$ where by*

convention $g(0) = 1$. Then

$$\int_{\text{Sym}^\bullet T} G \, de = \left(\sum_{a=0}^{\infty} g(a) q^a \right)^{e(T)}.$$

This lemma is a consequence of the fact that symmetric products define a pre-lambda ring structure on the Grothendieck group of varieties and the Euler characteristic homomorphism is compatible with that structure. An elementary proof is given in [5].

Applying Lemma 4.2 to Proposition 4.1 and combining with equations (2), (3), and (4) and we see that

$$(5) \quad q \widehat{\text{DT}}_0(X) = \frac{p}{(1-p)^2} \left(\sum_{a=0}^{\infty} F(a) q^a \right)^2 \cdot \prod_{m=1}^{\infty} (1 - q^m)^{-22}.$$

To finish the computation of $\widehat{\text{DT}}_0(X)$, we need to prove Proposition 4.1 and compute the series $\sum_a F(a) q^a$.

4.1. Proof of Proposition 4.1 and the computation of $\sum_a F(a) q^a$. The fiber $\rho^{-1}(\sum a_i y_i)$ parameterizes subschemes supported on $\widehat{X}_{C_0 \times E}$ which have fixed curve support

$$C_0 \times x_0 \cup_i \{y_i\} \times E$$

where the multiplicity of the subscheme along $\{y_i\} \times E$ is a_i . Such a subscheme is *uniquely* determined³ by its restriction to the formal neighborhoods $\widehat{X}_{\{y_i\} \times E}$ and their complement U in $\widehat{X}_{C_0 \times E}$. The resulting stratification leads to a product decomposition for the Grothendieck group valued power series $\rho^{-1}(\sum a_i y_i)$ giving the product formula in Proposition 4.1. The factor $p(1-p)^{-2}$ comes from the contribution of U and it is the series for the Hilbert scheme of subschemes of $\widehat{X}_{C_0 \times E}$ with fixed curve support $C_0 \times x_0$ (no curves in the E direction). The moduli for this Hilbert scheme comes from floating points and embedded points (see [5] for details).

The series $F(a)$ is given by

$$F(a) = (1-p) \cdot e \left(\text{Hilb}^{0,a,\bullet} \left(\widehat{X}_{\{y_i\} \times E} \right) \right)$$

where

$$\text{Hilb}^{0,a,\bullet} \left(\widehat{X}_{\{y_i\} \times E} \right) \subset \text{Hilb}^{0,a,\bullet}(X)$$

is the locus parameterizing subschemes Z whose curve support is given by the union of $C_0 \times \{x_0\}$ and an a -fold thickening of $\{y_i\} \times E$ and such that all embedded points of Z are supported on $\widehat{X}_{\{y_i\} \times E}$. The prefactor $(1-p)$ comes from the contribution of the complement U : the overall contribution of U is given by $p(1-p)^{-2+l}$ where l is the number of y_i 's and so we have redistributed the l copies of $(1-p)$ into the $F(a_i)$ factors.

Since

$$\widehat{X}_{\{y_i\} \times E} \cong \text{Spec}(\mathbb{C}[[u, v]]) \times E,$$

³This follows from fpqc descent since the set U and the sets $\widehat{X}_{\{y_i\} \times E}$ form a fpqc cover. Since $C_0 \times x_0$ is reduced there are no conditions on the overlaps of the cover. Thus the subscheme is *uniquely* determined by its restriction to the cover.

we get an action of $(\mathbb{C}^*)^2$ on the corresponding Hilbert scheme. Only the $(\mathbb{C}^*)^2$ fixed points contribute to the Euler characteristic so

$$\begin{aligned} F(a) &= (1-p) \cdot e \left(\text{Hilb}^{0,a,\bullet} \left(\widehat{X}_{\{y_i\} \times E} \right)^{(\mathbb{C}^*)^2} \right) \\ &= (1-p) \sum_{\alpha \vdash a} e \left(\text{Hilb}^{0,\alpha,\bullet} \left(\widehat{X}_{\{y_i\} \times E} \right) \right) \end{aligned}$$

where $\text{Hilb}^{0,\alpha,\bullet} \left(\widehat{X}_{\{y_i\} \times E} \right)$ parameterizes subschemes whose curve component is the *unique* curve given by the union of $C_0 \times \{x_0\}$ and $Z_\alpha \times E$ where $Z_\alpha \subset \text{Spec}(\mathbb{C}[[u, v]])$ is the length a subscheme given by the monomial ideal determined⁴ by the partition $\alpha \vdash a$.

To compute $e \left(\text{Hilb}^{0,\alpha,\bullet} \left(\widehat{X}_{\{y_i\} \times E} \right) \right)$ we can now integrate over the fibers of the constructible morphism

$$\sigma : \text{Hilb}^{0,\alpha,\bullet} \left(\widehat{X}_{\{y_i\} \times E} \right) \rightarrow \text{Sym}^\bullet E$$

which is defined by recording the length and locations of the embedded points. We thus get

$$\int_{\text{Hilb}^{0,\alpha,\bullet} \left(\widehat{X}_{\{y_i\} \times E} \right)} de = \int_{\text{Sym}^\bullet E} \sigma_*(1) de.$$

The constructible function $\sigma_*(1)$ is a product of local contributions which only depend on the length of the embedded point and whether or not the location of the embedded point is x_0 or not (recall that x_0 is where the curve $C_0 \times \{x_0\}$ is attached to the curve $Z_\alpha \times E$). Writing the series for the local contributions at x_0 and at the general point as $V_{\emptyset(1)\alpha}(p)$ and $V_{\emptyset\emptyset\alpha}(p)$ respectively, and applying Lemma 4.2 we get

$$\begin{aligned} \int_{\text{Sym}^\bullet E} \sigma_*(1) de &= (V_{\emptyset(1)\alpha}(p)) \cdot (V_{\emptyset\emptyset\alpha}(p))^{e(E-x_0)} \\ &= \frac{V_{\emptyset(1)\alpha}(p)}{V_{\emptyset\emptyset\alpha}(p)}. \end{aligned}$$

The above naming of the local contributions is not accidental — the generating functions for the contributions are given by the topological vertex. In general, the topological vertex $V_{\mu_1\mu_2\mu_3}(p)$ can be defined as the generating function of the Euler characteristics of the Hilbert schemes $\text{Hilb}^n \left(\widehat{\mathbb{C}}_0^3, \{\mu_1, \mu_2, \mu_3\} \right)$, which by definition parameterize subschemes of \mathbb{C}^3 given by adding at the origin a length n embedded point to the fixed curve $Z_{\mu_1} \cup Z_{\mu_2} \cup Z_{\mu_3}$. Here Z_{μ_i} is supported on the i th coordinate axis and given by the monomial ideal determined by the partition μ_i in the transverse directions. Because $(\mathbb{C}^*)^3$ acts on these Hilbert schemes, their Euler characteristics can be computed by counting $(\mathbb{C}^*)^3$ fixed points, namely monomial ideals. This leads to the combinatorial interpretation of $V_{\mu_1\mu_2\mu_3}(p)$ — it is the generating function for the number of 3D partitions with asymptotic legs given by $\{\mu_1, \mu_2, \mu_3\}$.

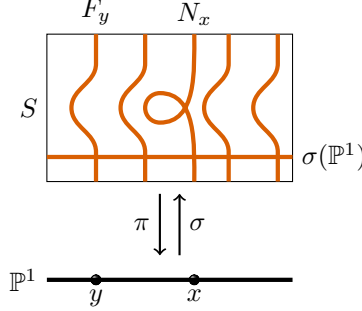
We thus get the following formula

$$\sum_{a=0}^{\infty} F(a) q^a = \sum_{\alpha} q^{|\alpha|} (1-p) \frac{V_{\emptyset(1)\alpha}(p)}{V_{\emptyset\emptyset\alpha}(p)}$$

which completes the proof of Proposition 4.1.

⁴i.e. identifying the partition α with its Ferrer's diagram $\alpha \subset (\mathbb{Z}_{\geq 0})^2$, the ideal of Z_α is generated by the monomials $u^i v^j$ where $(i, j) \notin \alpha$.

FIGURE 4. $\pi : S \rightarrow \mathbb{P}^1$ is an elliptic fibration with 24 nodal fibers and a section σ . Figure depicts a smooth fiber F_y over a point $y \in \mathbb{P}^1$ and a nodal fiber N_x over a point $x \in \mathbb{P}^1$.



Lemma 4.3. *The generating function for the universal series $F(a)$ is given by the following formula*

$$\sum_{a=0}^{\infty} F(a) q^a = \prod_{m=1}^{\infty} \frac{(1 - q^m)}{(1 - pq^m)(1 - p^{-1}q^m)}.$$

Proof. Using the Okounkov-Reshetikhin-Vafa formula for the vertex [11, eqn 3.20], the sum

$$\sum_{\alpha} q^{|\alpha|} (1 - p) \frac{V_{\emptyset(1)\alpha}(p)}{V_{\emptyset\emptyset\alpha}(p)}$$

can be expressed as the trace of a certain natural operator on Fock space. It can be evaluated explicitly by a theorem of Bloch-Okounkov [2, Thm 6.5]. The result is the product formula given by the lemma. See [6] for details. \square

Substituting the formula of the lemma into equation (5) we get

$$\widehat{\mathrm{DT}}_0(X) = \frac{pq^{-1}}{(1-p)^2} \prod_{m=1}^{\infty} (1 - q^m)^{-20} (1 - pq^m)^{-2} (1 - p^{-1}q^m)^{-2}$$

which proves the $g = 0$ formula in Theorem 2.4, assuming that we can show

$$\mathrm{DT}_0(X) = -\widehat{\mathrm{DT}}_0(X).$$

We will address this issue in section 6.

5. THE CASE OF $h = 1$.

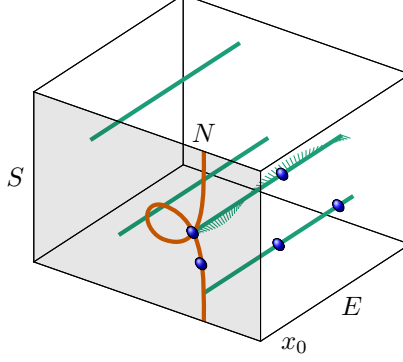
We now consider the case where S has a primitive curve class β with $\beta^2 = 0$. Such $K3$ surfaces are elliptically fibered with fiber class β . By our genericity assumption, we may assume that the elliptic fibration $\pi : S \rightarrow \mathbb{P}^1$ has 24 singular fibers, all of which are nodal, and we will further assume that the fibration has a section (see figure 4).

Recall that the Hilbert scheme decomposes into a disjoint union

$$\mathrm{Hilb}^{1,d,n}(X) = \mathrm{Hilb}_{\mathrm{vert}}^{1,d,n}(X) \sqcup \mathrm{Hilb}_{\mathrm{diag}}^{1,d,n}(X).$$

We can fix a slice for the E action on $\mathrm{Hilb}_{\mathrm{vert}}^{1,d,n}(X)$ by requiring that the unique vertical curve lies in $S \times \{x_0\}$. In the case where the subscheme has a diagonal curve, we require

FIGURE 5. A configuration which includes a thickened horizontal curve (green) attached to the node of a nodal vertical curve (orange). For the contribution to be non-zero, embedded points (blue) must occur along horizontal curves attached to the vertical curve or on the vertical curve.



that the diagonal curve intersects the slice $S \times \{x_0\}$ somewhere on the section. Denoting the above conditions with the subscript fixed, we get

$$\mathrm{Hilb}^{1,d,n}(X)/E \cong \mathrm{Hilb}_{\mathrm{vert},\mathrm{fixed}}^{1,d,n}(X) \sqcup \mathrm{Hilb}_{\mathrm{diag},\mathrm{fixed}}^{1,d,n}(X)$$

and so

$$q\widehat{\mathrm{DT}}_1(X) = e\left(\mathrm{Hilb}_{\mathrm{vert},\mathrm{fixed}}^{1,\bullet,\bullet}(X)\right) + e\left(\mathrm{Hilb}_{\mathrm{diag},\mathrm{fixed}}^{1,\bullet,\bullet}(X)\right).$$

We get a map

$$\tau : \mathrm{Hilb}_{\mathrm{vert},\mathrm{fixed}}^{1,\bullet,\bullet}(X) \rightarrow \mathbb{P}^1$$

induced by the elliptic fibration $S \rightarrow \mathbb{P}^1$ since each subscheme parameterized by $\mathrm{Hilb}_{\mathrm{vert},\mathrm{fixed}}^{1,\bullet,\bullet}(X)$ has a unique vertical curve which is a fiber curve. Let F_y denote the fiber of $S \rightarrow \mathbb{P}^1$ over y . Let

$$\mathrm{Hilb}_{F_y}^{1,d,n}(X) \subset \mathrm{Hilb}_{\mathrm{vert},\mathrm{fixed}}^{1,d,n}(X)$$

denote the sublocus which parameterizes subschemes whose unique vertical component is $F_y \times \{x_0\}$.

We will see below that the Euler characteristic of $\mathrm{Hilb}_{F_y}^{1,d,n}(X)$ only depends on the topological type of the fiber, i.e. whether it is smooth or nodal. We write a generic smooth fiber as F and any nodal fiber as N . Integrating over the fibers of τ , we get

$$e\left(\mathrm{Hilb}_{\mathrm{vert},\mathrm{fixed}}^{1,\bullet,\bullet}(X)\right) = -22e\left(\mathrm{Hilb}_F^{1,\bullet,\bullet}(X)\right) + 24e\left(\mathrm{Hilb}_N^{1,\bullet,\bullet}(X)\right)$$

where the -22 is $e(\mathbb{P}^1 - 24\mathrm{pts})$. See figure 5 for a depiction of a curve configuration corresponding to a point in $\mathrm{Hilb}_N^{1,\bullet,\bullet}(X)$.

The computation of $e\left(\mathrm{Hilb}_F^{1,\bullet,\bullet}(X)\right)$ and $e\left(\mathrm{Hilb}_N^{1,\bullet,\bullet}(X)\right)$ follows the same strategy as the computation of $e\left(\mathrm{Hilb}_{\mathrm{fixed}}^{0,\bullet,\bullet}(X)\right)$ done in section 4. We use the product decompositions

$$\mathrm{Hilb}_F^{1,\bullet,\bullet}(X) = \mathrm{Hilb}_F^{1,\bullet,\bullet}\left(\widehat{X}_{F \times E}\right) \cdot \mathrm{Hilb}^{1,\bullet,\bullet}(X - F \times E)$$

$$\mathrm{Hilb}_N^{1,\bullet,\bullet}(X) = \mathrm{Hilb}_N^{1,\bullet,\bullet}\left(\widehat{X}_{N \times E}\right) \cdot \mathrm{Hilb}^{1,\bullet,\bullet}(X - N \times E)$$

and we use the extra E actions on the second factors to deduce

$$\begin{aligned} e\left(\mathrm{Hilb}_F^{1,\bullet,\bullet}(X)\right) &= e\left(\mathrm{Hilb}_F^{1,\bullet,\bullet}\left(\widehat{X}_{F \times E}\right)\right) \cdot \prod_{m=1}^{\infty} (1 - q^m)^{-24} \\ e\left(\mathrm{Hilb}_N^{1,\bullet,\bullet}(X)\right) &= e\left(\mathrm{Hilb}_N^{1,\bullet,\bullet}\left(\widehat{X}_{N \times E}\right)\right) \cdot \prod_{m=1}^{\infty} (1 - q^m)^{-23} \end{aligned}$$

where $24 = e(S - F)$ and $23 = e(S - N)$.

Proceeding as we did in section 4, we use the maps

$$\begin{aligned} \rho : \mathrm{Hilb}_F^{1,\bullet,\bullet}\left(\widehat{X}_{F \times E}\right) &\rightarrow \mathrm{Sym}^\bullet(F) \\ \rho : \mathrm{Hilb}_N^{1,\bullet,\bullet}\left(\widehat{X}_{N \times E}\right) &\rightarrow \mathrm{Sym}^\bullet(N) \end{aligned}$$

which record the location and multiplicity of the horizontal components. The argument proceeds exactly as it did in section 4 with F and N playing the role of C_0 .

The result for the smooth fiber case is the following:

$$\begin{aligned} \int_{\mathrm{Hilb}_F^{1,\bullet,\bullet}\left(\widehat{X}_{F \times E}\right)} de &= \int_{\mathrm{Sym}^\bullet(F)} \rho_*(1) de \\ &= \left(p^{1/2}(1-p)^{-1}\right)^{e(F)} \cdot \left(\sum_{a=0}^{\infty} F(a)q^a\right)^{e(F)} \\ &= 1. \end{aligned}$$

This result comports with the heuristic that F acts on $\widehat{X}_{F \times E}$ and hence on $\mathrm{Hilb}_F^{1,\bullet,\bullet}\left(\widehat{X}_{F \times E}\right)$ and so the Euler characteristic is 0 except for the unique F -fixed subscheme, i.e. the subscheme consisting of just the curve $F \times \{x_0\}$ with no added horizontal components or embedded points. However, this is only a heuristic: F does *not* act algebraically on the formal neighborhood $\widehat{X}_{F \times E}$ since the elliptic fibration is not isotrivial⁵.

The situation for nodal fibers is a little different because of the presence of the nodal point $z \in N$. The constructible function $\rho_*(1)$, which is given by taking the Euler characteristic of the fibers of the map

$$\rho : \mathrm{Hilb}_N^{1,\bullet,\bullet}\left(\widehat{X}_{N \times E}\right) \rightarrow \mathrm{Sym}^\bullet N,$$

has the following form. Let y_1, \dots, y_l be non-singular points of N and let $z \in N$ be the nodal point. Then $\rho^{-1}(bz + \sum a_i y_i)$ parameterizes subschemes of X , supported on $\widehat{X}_{N \times E}$, which have fixed curve support

$$N \times \{x_0\} \cup \{z\} \times E \cup_i \{y_i\} \times E$$

where the multiplicity along $\{z\} \times E$ is b and the multiplicity along $\{y_i\} \times E$ is a_i . Such a subscheme is determined by its restriction to the formal neighborhoods $\widehat{X}_{\{z\} \times E}$, $\widehat{X}_{\{y_1\} \times E}, \dots, \widehat{X}_{\{y_l\} \times E}$ and their complement U . The contribution of the Euler characteristic of U is given by

$$(1-p)^{-e(N^\circ)} = (1-p)^l$$

⁵Conceivably, there might be a topological action of F on $\mathrm{Hilb}_F^{1,d,n}\left(\widehat{X}_{F \times E}\right)$ which would make this heuristic rigorous.

where $N^\circ = N - \{z, y_1, \dots, y_l\}$. Therefore we see that

$$\rho_*(1)(bz + \sum a_i y_i) = N(b) \prod_{i=1}^l F(a_i)$$

where $F(a)$ is as in section 4, and

$$N(b) = e \left(\text{Hilb}^{1,b,\bullet} \left(\widehat{X}_{\{z\} \times E} \right) \right)$$

where

$$\text{Hilb}^{1,b,n} \left(\widehat{X}_{\{z\} \times E} \right) \subset \text{Hilb}^{1,b,n}(X)$$

is the sublocus parameterizing subschemes Z whose curve support is given by the union of $N \times \{x_0\}$ and a b -fold thickening of $\{z\} \times E$ and such that all embedded points are supported on $\widehat{X}_{\{z\} \times E}$.

So pushing the integral to $\text{Sym}^\bullet N$ and applying lemma 4.2 we get

$$\begin{aligned} \int_{\text{Hilb}_N^{1,\bullet,\bullet}(\widehat{X}_{N \times E})} 1 \, de &= \int_{\text{Sym}^\bullet N} \rho_*(1) \, de \\ &= \int_{\text{Sym}^\bullet(N - \{z\})} \prod_i F(a_i) \, de \cdot \int_{\text{Sym}^\bullet(\{z\})} N(b) \, de \\ &= \left(\sum_{a=0}^{\infty} F(a) q^a \right)^{e(N - \{z\})} \cdot \left(\sum_{b=0}^{\infty} N(b) q^b \right). \end{aligned}$$

Note that $e(N - \{z\}) = 0$ so that the $F(a)$ term doesn't contribute.

We compute the $N(b)$ contribution by using the $(\mathbb{C}^*)^2$ action on

$$\widehat{X}_{\{z\} \times E} \cong \text{Spec}(\mathbb{C}[[u, v]]) \times E$$

and arguing as in section 4. We find

$$\begin{aligned} \sum_{b=0}^{\infty} N(b) q^b &= \sum_{b=0}^{\infty} e \left(\text{Hilb}^{1,b,\bullet} \left(\widehat{X}_{\{z\} \times E} \right) \right) q^b \\ &= \sum_{b=0}^{\infty} \sum_{\beta \vdash b} e \left(\text{Hilb}^{1,\beta,\bullet} \left(\widehat{X}_{\{z\} \times E} \right) \right) q^b \\ &= \sum_{\beta} q^{|\beta|} \frac{V_{(1)(1)\beta}(p)}{V_{\emptyset\emptyset\beta}(p)}. \end{aligned}$$

We see that fact that the curve N has a node is manifest in the term in the numerator: the vertex $V_{(1)(1)\beta}(p)$ is counting curve configurations which are locally monomial at the nodal point $\{z\} \times \{x_0\}$ where the curve is degree 1 along the two branches of the node and has the monomial thickening given by β along the E direction.

Putting this and the earlier computations together, we find that the total contribution of the components with vertical curves is given by the following:

$$\begin{aligned} e \left(\text{Hilb}_{\text{vert, fixed}}^{1,\bullet,\bullet}(X) \right) &= -22 \prod_{m=1}^{\infty} (1 - q^m)^{-24} + 24 \prod_{m=1}^{\infty} (1 - q^m)^{-23} \cdot \sum_{\beta} q^{|\beta|} \frac{V_{(1)(1)\beta}(p)}{V_{\emptyset\emptyset\beta}(p)} \\ &= 24 \prod_{m=1}^{\infty} (1 - q^m)^{-24} \left\{ \frac{1}{12} - 1 + \prod_{m=1}^{\infty} (1 - q^m) \sum_{\beta} q^{|\beta|} \frac{V_{(1)(1)\beta}(p)}{V_{\emptyset\emptyset\beta}(p)} \right\}. \end{aligned}$$

Proposition 5.1. *The following identity holds:*

$$\prod_{m=1}^{\infty} (1 - q^m) \sum_{\beta} q^{|\beta|} \frac{V_{(1)(1)\beta}(p)}{V_{\emptyset\emptyset\beta}(p)} = 1 + \frac{p}{(1-p)^2} + \sum_{d=1}^{\infty} \sum_{k|d} k(p^k + p^{-k})q^d.$$

Proof sketch: Using the Okounkov-Reshetikhin-Vafa formula for the topological vertex [11, Eqn 3.20], and some standard combinatorics, one can rewrite the left hand side of the above equation so that it is given in terms of Bloch-Okounkov's 2-point correlation function [2, Eqn 5.2]. Namely, one can show that it is given by $1 - F(t_1, t_2)$ in the limit where t_1 and t_2 approach p and p^{-1} respectively. The limit can be evaluated explicitly using [2, Thm 6.1] and this leads to the right hand side of the formula. Details can be found in [6].

Plugging in the proposition's formula into the previously obtained equation, we see that the non-diagonal contribution to $\widehat{DT}_1(X)$ is given as

$$e\left(\mathrm{Hilb}_{\mathrm{vert}, \mathrm{fixed}}^{1, \bullet, \bullet}(X)\right) = 24 \prod_{m=1}^{\infty} (1 - q^m)^{-24} \left\{ \frac{1}{12} + \frac{p}{(1-p)^2} + \sum_{d=1}^{\infty} \sum_{k|d} k(p^k + p^{-k})q^d \right\}$$

5.1. Diagonal contributions. To finish our computation of $\widehat{DT}_1(X)$, it remains to compute $e\left(\mathrm{Hilb}_{\mathrm{diag}, \mathrm{fixed}}^{1, \bullet, \bullet}(X)\right)$.

Let $C \subset X$ be a diagonal curve. The projections onto the factors of $X = S \times E$ induce maps

$$\begin{aligned} p_S : C &\rightarrow F_y \\ p_E : C &\rightarrow E \end{aligned}$$

where F_y is a fiber curve, and the maps have degree 1 and some $d > 0$ respectively. F_y cannot be a nodal fiber since then C would have geometric genus 0 and consequently it would not admit a non-constant map to E . The above maps induce a map

$$f : F_y \rightarrow E$$

which must be unramified by the Riemann-Hurwitz formula. Thus the diagonal curve C is contained in the surface $F_y \times E$ and is given by the graph of the map f . Recall that we fixed a slice for the E action on $\mathrm{Hilb}_{\mathrm{diag}}^{1, d, n}(X)$ by requiring that the diagonal curve meets S_{x_0} at the section; this is equivalent to requiring that $f(s) = x_0$ where $s \in F_y$ is the section point on F_y . Up to automorphisms, such a map f must be a group homomorphism of the corresponding elliptic curves. Assuming that E is generic, so that the only non-trivial automorphism is given by $x \mapsto -x$, we see that every diagonal curve (with the fixed condition) is of the form

$$\{(z, f(z)) \in F_y \times E\} \text{ or } \{(z, -f(z)) \in F_y \times E\}$$

where $f : F_y \rightarrow E$ is a group homomorphism.

The number of group homomorphisms of degree d to a fixed elliptic curve E is given by $\sum_{k|d} k$. This classical fact can be seen by counting index d sublattices of $\mathbb{Z} \oplus \mathbb{Z}$. For each such cover, $F \rightarrow E$, the domain elliptic curve will occur exactly 24 times in the fibration $S \rightarrow \mathbb{P}^1$. So we find that the total number of diagonal curves having degree d in the E direction is

$$2 \cdot 24 \sum_{k|d} k.$$

Each such diagonal curve can be accompanied by horizontal curves (with thickenings) as well as embedded points. The contribution of these components of the Hilbert scheme is computed in exactly the same way as the contribution of the curves with a smooth vertical component F . Recall that $e\left(\text{Hilb}_F^{1,\bullet,\bullet}(X)\right) = \prod_{m=1}^{\infty} (1 - q^m)^{-24}$. Taking into account the degree of the diagonal curves, we thus find

$$e\left(\text{Hilb}_{\text{diag, fixed}}^{1,\bullet,\bullet}(X)\right) = \prod_{m=1}^{\infty} (1 - q^m)^{-24} \left(2 \cdot 24 \cdot \sum_{d=1}^{\infty} \sum_{k|d} k q^k \right).$$

Finally, adding the vertical and diagonal contributions together we arrive at

$$\widehat{\text{DT}}_1(X) = 24q^{-1} \prod_{m=1}^{\infty} (1 - q^m)^{-24} \left\{ \frac{1}{12} + \frac{p}{(1-p)^2} + \sum_{d=1}^{\infty} \sum_{k|d} k(p^k + p^{-k} + 2)q^d \right\}.$$

Note that this formula is off from the desired formula for $\text{DT}_{h=1}(X)$ by an overall minus sign and a minus sign on the 2. In fact we will see in section 6 that due to the Behrend function, the contribution of the diagonal components carry the opposite sign of the contribution of the vertical components. Denoting the contribution to $\widehat{\text{DT}}_1(X)$ coming from $\text{Hilb}_{\text{vert, fixed}}^{1,\bullet,\bullet}(X)$ and from $\text{Hilb}_{\text{diag, fixed}}^{1,\bullet,\bullet}(X)$ by $\widehat{\text{DT}}_{1,\text{vert}}(X)$ and $\widehat{\text{DT}}_{1,\text{diag}}(X)$ respectively, we find that we need to show

$$\text{DT}_1(X) = -\widehat{\text{DT}}_{1,\text{vert}}(X) + \widehat{\text{DT}}_{1,\text{diag}}(X).$$

6. PUTTING IN THE BEHREND FUNCTION

6.1. Overview. Our general strategy for computing $\widehat{\text{DT}}(X)$, the unweighted Euler characteristics of the Hilbert schemes, utilized the following general scheme.

- (1) Using the geometric support of curves (and/or points) of the subschemes, we stratified $\text{Hilb}(X)$ such that the strata could be written as products of simpler Hilbert schemes.
- (2) We utilized actions of \mathbb{C}^* or E which could be defined on individual factors in the stratification to discard strata not fixed by the action and restrict to fixed points.
- (3) We found that some strata were parameterized by symmetric products, and we pushed forward the Euler characteristic computation to the symmetric products where we used Lemma 4.2.
- (4) After possibly iterating steps (1)–(3), we reduced the computation to counting discrete subscheme configurations, namely those which are given formally locally by monomial ideals. These we counted with the topological vertex.

The Behrend function is not compatible in any naïve way with the Grothendieck ring (the Grothendieck ring is insensitive to singularities and non-reduced structures), but it is compatible with a modification of the Grothendieck ring, namely the Hall algebra. The main thing we need to do to make our strategy compatible with the Behrend function weighted Euler characteristics is to show that our various decompositions of the Hilbert scheme strata (step (1) above) can be written in terms of products in the Hall algebra. Finally, we will need to compute the value of the Behrend function at the locally monomial schemes, which we do using a computation of Maulik-Nekrasov-Okounkov-Pandharipande [8].

We use Joyce and Song's theory [7], particularly Bridgeland's approach to Donaldson-Thomas invariants via Joyce-Song theory [4, 3]. We outline this approach briefly here.

For an arbitrary Calabi-Yau threefold X , let $\text{Coh}_{\leq 1}(X) \subset \text{Coh}(X)$ be the subcategory of coherent sheaves on X which are supported in dimension one or less. Let

$$\mathcal{M} = \bigsqcup_{\beta, n} \mathcal{M}_{\beta, n}$$

be the moduli stack of objects in $\text{Coh}_{\leq 1}(X)$ where $\mathcal{M}_{\beta, n}$ is the component whose objects have Chern character $(0, 0, \beta, n) \in H^{ev}(X)$. Let

$$\text{Hall}(X)$$

be the relative Grothendieck group of stacks over \mathcal{M} . We equip $\text{Hall}(X)$ with an associate product (the Hall product) which is defined in terms of extensions: let

$$[f : V \rightarrow \mathcal{M}], [g : U \rightarrow \mathcal{M}] \in \text{Hall}(X),$$

then the Hall product is given by

$$[f : V \rightarrow \mathcal{M}] \star [g : U \rightarrow \mathcal{M}] = [h : W \rightarrow \mathcal{M}]$$

where W is the stack whose objects are triples (v, u, E) where $v \in V$, $u \in U$ and E is an extension of $g(u)$ by $f(v)$. The map h is given by $(v, u, E) \mapsto E$. The Hall product is associate but typically non-commutative.

Let $H_{\text{reg}}(X) \subset \text{Hall}(X)$ be the subalgebra⁶ generated by elements of the form $[f : H \rightarrow \mathcal{M}]$ where H is a scheme. We define the Joyce-Song integration map by

$$\begin{aligned} \Phi : H_{\text{reg}}(X) &\rightarrow \mathbb{Z}((p))[[q]] \\ [f : H \rightarrow \mathcal{M}] &\mapsto \sum_{\beta, n} p^n q^\beta \int_{f^{-1}(\mathcal{M}_{\beta, n})} f^*(\nu_{\mathcal{M}}) de \end{aligned}$$

where $\nu_{\mathcal{M}} : \mathcal{M} \rightarrow \mathbb{Z}$ is the Behrend function.

Theorem 6.1 (Joyce-Song). *The map Φ is an algebra homomorphism⁷.*

The Hilbert schemes define an element

$$[f : \text{Hilb}(X) \rightarrow \mathcal{M}] \in H_{\text{reg}}(X)$$

where the map f is given by $Z \mapsto \mathcal{O}_Z$. Bridgeland shows [3, Thm 3.1] that the Behrend function on the Hilbert scheme differs from the Behrend function on \mathcal{M} by $(-1)^n$:

$$(6) \quad \nu_{\text{Hilb}^{\beta, n}(X)} = (-1)^n f^*(\nu_{\mathcal{M}_{\beta, n}}).$$

Thus we see that since we defined the Donaldson-Thomas partition function $DT(X)$ by

$$DT(X) = \sum_{\beta, n} DT_{\beta, n}(X) (-p)^n q^\beta,$$

then

$$DT(X) = \Phi([f : \text{Hilb}(X) \rightarrow \mathcal{M}]).$$

We now return to the case of $X = S \times E$ where S has a fixed primitive curve class β of square $2h - 2$. To make the Hall algebra machinery compatible with our bullet convention, we formally label the component $\mathcal{M}_{\beta + dE, n}$ by the monomial $p^n q^d$. That is, we regard elements in $\text{Hall}(X)$ as formal power series in p and q whose $p^n q^d$ coefficient is in the

⁶ $H_{\text{reg}}(X)$ is closed under the Hall product when viewed as a $K_0(\text{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ algebra.

⁷Even more is true: there is a Poisson structure on $H_{\text{reg}}(X)/(\mathbb{L} - 1)$ such that Φ is a Poisson algebra homomorphism. We won't need this however.

relative Grothendieck group of stacks over $\mathcal{M}_{\beta+dE,n}$. The integration map is then given by

$$\Phi : \sum_{d,n} [f : H_{\beta+dE,n} \rightarrow \mathcal{M}_{\beta+dE,n}] p^n q^d \mapsto \sum_{d,n} \left(\int_{H_{\beta+dE,n}} f^*(\nu_{\mathcal{M}_{\beta+dE,n}}) de \right) p^n q^d.$$

Recall that

$$\text{Hilb}^{h,n,d}(X) \cong \text{Hilb}_{\text{vert}}^{h,n,d}(X) \sqcup \text{Hilb}_{\text{diag}}^{h,n,d}(X)$$

and that

$$\text{Hilb}^{h,n,d}(X)/E \cong \text{Hilb}_{\text{vert, fixed}}^{h,n,d}(X) \sqcup \text{Hilb}_{\text{diag, fixed}}^{h,n,d}(X)$$

where we have identified the quotient on the left with a slice for the action defined geometrically via the “fixed” condition.

We adapt the bullet convention to the Hall algebra so that, for example, $\text{Hilb}_{\text{vert, fixed}}^{h, \bullet, \bullet}(X)$ denotes the element

$$\sum_{d,n} \left[\text{Hilb}_{\text{vert, fixed}}^{h,d,n}(X) \rightarrow \mathcal{M}_{\beta+dE,n} \right] q^d p^n$$

in $\text{Hall}(X)$.

In the Hall algebra language, we find that the partition functions for the reduced Donaldson-Thomas invariants of X are given by

$$q \cdot \text{DT}_h(X) = -\Phi \left(\text{Hilb}_{\text{fixed}}^{h, \bullet, \bullet}(X) \right).$$

The minus sign is due to the fact that the Behrend function on the quotient differs from the Behrend function of the slice by a factor of -1 since the map to the quotient is smooth of relative dimension 1.

To make our previous arguments work for the Behrend function weighted Euler characteristics, we need to check that every time we wrote a stratification of the Hilbert scheme into strata written as products of simpler Hilbert schemes, that product is induced by the Hall product of the underlying structure sheaves.

For example, the product in the Grothendieck group given by equation (1) was induced by decomposing a subscheme into components supported in the formal neighborhood $\widehat{X}_{C_0 \times E}$ and components supported on its complement. The same equation holds in the Hall algebra since if a subscheme Z is given as the disjoint union $Z = Z_1 \sqcup Z_2$ of subschemes Z_1 and Z_2 , then \mathcal{O}_Z is the unique⁸ extension of \mathcal{O}_{Z_1} by \mathcal{O}_{Z_2} given by $\mathcal{O}_{Z_1} \oplus \mathcal{O}_{Z_2}$.

We also obtained product decompositions of Hilbert schemes arising from subscheme configurations which have some fixed subscheme. For example, the subschemes Z parameterized by $\text{Hilb}_{\text{fixed}}^{0,d,n}(\widehat{X}_{C_0 \times E})$ (depicted in Figure 3) always have $C_0 = C_0 \times \{x_0\}$ as a subscheme. The exact sequence

$$0 \rightarrow I_{C_0/Z} \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{C_0} \rightarrow 0$$

induces the product in the Hall algebra

$$\text{Hilb}_{\text{fixed}}^{0, \bullet, \bullet}(\widehat{X}_{C_0 \times E}) = I^{0, \bullet, \bullet}(\widehat{X}_{C_0 \times E}) \star \text{Hilb}_{\text{fixed}}^{0,0,0}(X)$$

⁸It is perhaps worth remarking that automorphisms are not an issue here. Elements in the Hilbert scheme can be viewed as pairs $\mathcal{O}_X \rightarrow \mathcal{O}_Z$ with the map $\text{Hilb}(X) \rightarrow \mathcal{M}$ given by $[\mathcal{O}_X \rightarrow \mathcal{O}_Z] \mapsto \mathcal{O}_Z$. While the sheaves \mathcal{O}_{Z_1} , \mathcal{O}_{Z_2} , and $\mathcal{O}_{Z_1} \oplus \mathcal{O}_{Z_2}$ can have complicated automorphism groups, the objects $[\mathcal{O}_X \rightarrow \mathcal{O}_{Z_1}]$, $[\mathcal{O}_X \rightarrow \mathcal{O}_{Z_2}]$, and their Hall product $[\mathcal{O}_X \rightarrow \mathcal{O}_{Z_1} \oplus \mathcal{O}_{Z_2}]$ are all automorphism free.

where $\text{Hilb}_{\text{fixed}}^{0,0,0}(X)$ is a single point (labelled by $p^1 q^0$) representing the subscheme C_0 and $I^{0,\bullet,\bullet}(\widehat{X}_{C_0 \times E})$ parameterizes the ideal sheaves $I_{C_0/Z}$ of C_0 in the (varying) subschemes Z . The sheaves $I_{C_0/Z}$ are supported on a collection of formal neighborhoods of the form $\widehat{X}_{\{y_i \times E\}}$ and their complement $U \subset \widehat{X}_{C_0 \times E}$ and so we get a direct sum

$$I_{C_0/Z} = I_{C_0/Z}|_U \oplus I_{C_0/Z}|_{\widehat{X}_{\{y_i\} \times E}}$$

which induces a product in the Hall algebra. This product is the same as the product in the Grothendieck group that we used in the beginning of § 4.1.

A little later in the same section, we did something similar: we decomposed

$$\text{Hilb}^{0,\alpha,\bullet}(\widehat{X}_{\{y_i\} \times E})$$

into products according to the support of the embedded points. Recall that α is a partition and $\text{Hilb}^{0,\alpha,\bullet}(\widehat{X}_{\{y_i\} \times E})$ parameterizes subschemes whose curve component is the unique curve given by

$$C_\alpha = C_0 \times \{x_0\} \cup Z_\alpha \times E.$$

Thus the ideal sheaf $I_{C_\alpha/Z}$ given by

$$0 \rightarrow I_{C_\alpha/Z} \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{C_\alpha} \rightarrow 0$$

is supported on points. Similarly to the previous case, the decomposition $I_{C_\alpha/Z}$ into a direct sum of sheaves supported on a single point induces a product in the Hall algebra which is the same as the product in the Grothendieck group that we used in section 4.1, namely the products induced by the stratification in $\text{Sym}^\bullet E$.

Thus we've seen that the stratification and product decompositions that we did in the Grothendieck group can be carried out completely in the Hall algebra. We also employed actions of $(\mathbb{C}^*)^2$, $(\mathbb{C}^*)^3$, and E on various strata. These actions are compatible with the Behrend function since they arise from actions on the underlying geometry (e.g. formal neighborhoods of points or horizontal curves).

We are thus reduced to computing the contribution of subschemes which are formally locally monomial. In the Grothendieck group computation, these were each counted with weight 1; we now need to count them weighted by the value of the Behrend function at the corresponding point in the Hilbert scheme.

Proposition 6.2.

- The value of the Behrend function of $\text{Hilb}_{\text{fixed}}^{0,d,n}(X)$ at a subscheme given formally locally by monomial ideals is given by $(-1)^{n+1}$.
- The value of the Behrend function of $\text{Hilb}_{\text{vert, fixed}}^{1,d,n}(X)$ at a subscheme given formally locally by monomial ideals is given by $(-1)^{n+1}$.
- The value of the Behrend function of $\text{Hilb}_{\text{diag, fixed}}^{1,d,n}(X)$ at a subscheme given formally locally by monomial ideals is given by $(-1)^n$.

As an immediate corollary, we get the desired relations between the DT and the $\widehat{\text{DT}}$ partition functions:

Corollary 6.3.

$$\begin{aligned} \text{DT}_0(X) &= -\widehat{\text{DT}}_0(X), \\ \text{DT}_1(X) &= -\widehat{\text{DT}}_{1,\text{vert}}(X) + \widehat{\text{DT}}_{1,\text{diag}}(X). \end{aligned}$$

So to complete the proof of our main theorem, Theorem 2.4, it remains only to prove Proposition 6.2.

6.2. Proof of Proposition 6.2. We begin by reducing the computation to the case of locally monomial subschemes without embedded points.

Let $Z \subset X$ be a subscheme which is formally locally given by monomial ideals. Let $Z^\circ \subset Z$ be the maximal Cohen-Macaulay subscheme in Z . It has the property that the ideal sheaf $I_{Z^\circ/Z}$ is supported at a collection of points $\{p_1, \dots, p_k\}$. Then the exact sequence

$$0 \rightarrow I_{Z^\circ/Z} \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{Z^\circ} \rightarrow 0$$

and the splitting

$$I_{Z^\circ/Z} = \bigoplus_{i=1}^k (I_{Z^\circ/Z})|_{p_i}$$

give rise to products in the Hall algebra. Consequently, the values of $\nu_{\mathcal{M}}$, the Behrend function on the moduli stack of sheaves, obeys

$$(7) \quad \nu_{\mathcal{M}}(\mathcal{O}_Z) = \nu_{\mathcal{M}}(\mathcal{O}_{Z^\circ}) \prod_{i=1}^k \nu_{\mathcal{M}}(I_{Z^\circ/Z}|_{p_i}).$$

We will determine the value of $\nu_{\mathcal{M}}(I_{Z^\circ/Z}|_{p_i})$ using the computation of Maulik-Nekrasov-Okounkov-Pandharipande (MNOP) [8] who computed the value of the Behrend function of the Hilbert scheme of a toric Calabi-Yau threefold at torus invariant subschemes.

Lemma 6.4. *Let $I_{Z^\circ/Z}|_{p_i}$ be as above, then $\nu_{\mathcal{M}}(I_{Z^\circ/Z}|_{p_i}) = 1$.*

Proof. Let W be a toric Calabi-Yau threefold and let $Y \subset W$ be a torus invariant, 1-dimensional subscheme. Let $\nu_{\text{Hilb}(W)}$ be the Behrend function on $\text{Hilb}(W)$. MNOP determines an explicit formula for $\nu_{\text{Hilb}(W)}(Y)$ in terms of toric data. Comparing their formula for Y and for $Y^\circ \subset Y$, the maximal Cohen-Macaulay subscheme, their result gives

$$\nu_{\text{Hilb}(W)}(Y) = (-1)^{\chi(I_{Y^\circ/Y})} \cdot \nu_{\text{Hilb}(W)}(Y^\circ).$$

Let $\nu_{\mathcal{M}(W)}$ be the Behrend function on $\mathcal{M}(W)$, the moduli stack of sheaves on W . Then using equation (6) and simplifying we see

$$\nu_{\mathcal{M}(W)}(\mathcal{O}_Y) = \nu_{\mathcal{M}(W)}(\mathcal{O}_{Y^\circ}).$$

On the other hand, the exact sequence

$$0 \rightarrow I_{Y^\circ/Y} \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{Y^\circ} \rightarrow 0$$

gives a product in the Hall algebra, applying the integration map, implies that

$$\nu_{\mathcal{M}(W)}(\mathcal{O}_Y) = \nu_{\mathcal{M}(W)}(I_{Y^\circ/Y}) \cdot \nu_{\mathcal{M}(W)}(\mathcal{O}_{Y^\circ})$$

and so we conclude that

$$\nu_{\mathcal{M}(Y)}(I_{Y^\circ/Y}) = 1$$

for any Y .

Returning to $Z \subset X$, we know that $I_{Z^\circ/Z}|_{p_i}$ is supported in the formal neighborhood $\widehat{X}_{p_i} \cong \text{Spec } \mathbb{C}[[x_1, x_2, x_3]]$ and is torus invariant. The formal neighborhood of the moduli stack $\mathcal{M}(X)$ at the point $[I_{Z^\circ/Z}|_{p_i}] \in \mathcal{M}(X)$ is isomorphic to a formal neighborhood of $[I_{Y^\circ/Y}] \in \mathcal{M}(Y)$ for some appropriately chosen toric $Y^\circ \subset Y \subset W$. Since the Behrend function at a point is determined by a formal neighborhood of that point, we conclude that

$$\nu_{\mathcal{M}(X)}(I_{Z^\circ/Z}|_{p_i}) = \nu_{\mathcal{M}(Y)}(I_{Y^\circ/Y}) = 1.$$

□

Applying the lemma to equation (7) and using the equation (6) we get

$$(-1)^{\chi(\mathcal{O}_Z)} \nu_{\text{Hilb}(X)}(Z) = (-1)^{\chi(\mathcal{O}_{Z^\circ})} \nu_{\text{Hilb}(X)}(Z^\circ).$$

The proof of Proposition 6.2 is thus reduced to determining the value of $\nu_{\text{Hilb}(X)}$ at subschemes $Z^\circ \subset X$ which are Cohen-Macaulay and formally locally monomial. This is done in the following:

Lemma 6.5. *There is a non-singular open set in $\text{Hilb}_{\text{fixed}}^{0,d,n}(X)$ of dimension $2d + n - 1$ which contains all the Cohen-Macaulay formally locally monomial subschemes. Similarly, there are non-singular open sets in $\text{Hilb}_{\text{fixed,vert}}^{1,d,n}(X)$ and in $\text{Hilb}_{\text{fixed,diag}}^{1,d,n}(X)$ of dimension $2d + n + 1$ and $2d + n$ respectively which contain all the Cohen-Macaulay formally locally monomial subschemes. Consequently, the value of the Behrend function on $\text{Hilb}_{\text{fixed}}^{0,d,n}(X)$, $\text{Hilb}_{\text{fixed,vert}}^{1,d,n}(X)$, and $\text{Hilb}_{\text{fixed,diag}}^{1,d,n}(X)$ at such subschemes is $(-1)^{n+1}$, $(-1)^{n+1}$, and $(-1)^n$ respectively.*

Proof. Consider the locus of points in $\text{Hilb}_{\text{fixed}}^{0,d,n}(X)$ parameterizing subschemes of the form

$$Z \times E \cup C_0 \times x_0$$

where $Z \subset S$ is a zero dimensional subscheme of length d . Such subschemes will necessarily satisfy the condition that $Z \cap C_0$ has length $1 - n$. The locus of such subschemes can thus be identified with locally closed subset of $\text{Hilb}^d(S)$ given by the condition that $Z \cap C_0$ has length $1 - n$. This subset is smooth and of codimension $1 - n$, hence dimension $2d + n - 1$. Finally, the locus of such subschemes is open in $\text{Hilb}_{\text{fixed}}^{0,d,n}(X)$ as any deformation of such a subscheme is a subscheme of the same type. A similar argument works for the other two cases with slightly different dimension counts. \square

7. PROSPECTS FOR $h > 1$

Our strategy can be applied to the case of computing $\text{DT}_h(X)$ for $h > 1$ although some new issues and complexities arises. Our method is predicated on two main things:

- (1) Having a detailed understanding of the possible curve support of subschemes in the class $\beta_h + dE$.
- (2) Having the singularities of the curves be formally locally toric so that vertex methods can be applied.

Addressing issue (1) grows increasingly difficult as h gets larger. For relatively small values of h , one has a pretty explicit understanding of the curves in the linear system of β_h . To address (1) fully also requires understanding “diagonal” curves. This amounts to solving the following interesting enumerative question about $K3$ surfaces:

Question 7.1. *Given a $K3$ surface with a irreducible curve class β of square $2h - 2$, how many curves of geometric genus g are in the class β which admit a degree d map to a (fixed) elliptic curve E ?*

Note that genus g curves on a $K3$ surface always move in an g dimensional family, and the dimension of genus g curves admitting a degree d map to an elliptic curve E is $2g - 3$ (independent of d) and thus is codimension g in \overline{M}_g . Therefore this is a dimension zero problem.

Addressing issue (2) requires some new ideas. Starting at $h = 2$, one must confront curves with singularities worse than nodes. For small h , one should be able to finesse around this issue. For example, for $h = 2$, one will need the contribution of a curve in $K3$ with a cusp, with a d -fold thickening of E attached at the cusp. This is not locally

toric and so its contribution cannot be computed using the vertex methods that we used for locally toric subschemes. However, this contribution can be fully determined from the $h = 1$ results as follows. One redoes the $h = 1$ computation using an elliptically fibered $K3$ which has a cuspidal singular fiber. This will enable one to reverse engineer the cusp contribution which one can then apply to compute the $h = 2$ case fully. However, it isn't clear how far one can get with this inductive sort of strategy.

A more satisfying way to handle a contribution from arbitrary surface singularity would be to relate this contribution to the knot invariants of the link of the singularities. This would be in keeping with the work of Shende and Oblomkov [10] although it doesn't appear that their results direct apply.

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